Superposition of chaotic processes with convergence to Lévy's stable law

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We construct a family of chaotic dynamical systems with explicit distributions with broad tails, which always violate the central limit theorem. In particular, we show that the superposition of many statistically independent, identically distributed random variables obeying such a chaotic process converge in density to Lévy's stable laws in a full range of index parameters in a unified manner. The theory related to the connection between deterministic chaos and non-Gaussian distributions gives us a systematic view of the purely mechanical generation of Lévy's stable laws. [S1063-651X(98)08408-6]

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Lévy's stable laws are the most famous class of distributions violating the central limit theorem (CLT). These arise in such diverse fields as astronomy, physics, biology, economics, and communication engineering, under broad conditions [1–4]. Our primary interest here is to elucidate the mechanism by which Lévy's stable laws are generated. In the 1980s, several studies clarified the relation between Levy's-law-like broad distributions and intermittent periodic mapping and anomalous diffusion [5,6]. Random-walk models [7] and combinations of several random number generators [8,9] are also utilized to generate Lévy's stable laws. However, the approximations made or the nondeterministic nature of the models themselves, or their generation methods are only applicable to a special class of Lévy's stable laws, because few examples of Lévy's stable distributions are explicitly known. The purpose of the present paper is to present an explicit implementation for exact and purely mechanical generation of stable laws with arbitrary indices using concrete chaotic dynamical systems. Let us consider a one-dimensional dynamical system

$$X_{n+1}=\frac{1}{2}\left(X_n-\frac{1}{X_n}\right)\equiv f(X_n) \tag{1}$$

on the infinite support $(-\infty,+\infty)$. Note that for this mapping $f(X)$ can be seen to be the doubling formula of $-\cot(\theta)$ as $-\cot(2\theta)=f[-\cot(\theta)]$. Thus the system has the exact solution $X_n=-\cot((\pi/2)2^n\theta_0)$. Using a diffeomorphism $x\equiv\phi^{-1}(\theta)=-[1/\tan((\pi/2)\theta)]$ of $\theta\in[0,2]$ into $]-\infty,+\infty[$, we derive the piecewise-linear map $g^{(2)}(\theta)=\phi\circ f\circ\phi^{-1}(\theta)$ as

$$g^{(2)}(\theta)=2\theta,\quad \theta\in[0,1),$$
$$g^{(2)}(\theta)=2\theta-2,\quad \theta\in[1,2]. \tag{2}$$

Because map (2) has the mixing property (and thus is clearly ergodic) and preserves the Lebesgue measure $\frac{1}{2}d\theta$ of $[0,2]$, map f preserves the measure

$$\mu(dx)=\rho(x)dx=\frac{1}{2}\frac{d\phi(x)}{dx}dx=\frac{dx}{\pi(1+x^2)}. \tag{3}$$

This is an explanation of the mechanical origin of the Cauchy distribution (3). Note that the Cauchy distribution is a simple case of a Lévy stable distribution with the characteristic $\alpha=1$. Measure (3) is absolutely continuous with respect to the Lebesgue measure, which implies that the Kolmogorov-Sinai entropy $h(\mu)$ is equivalent to the Lyapunov exponent of $\ln 2$ from the Pesin identity; the measure is a physical one in the sense that, for almost all initial conditions x_0, the time averages $\lim_{n\to\infty}(1/n)\Sigma_{i=0}^{n-1}\delta(x-x_i)$ reproduce the invariant measure $\mu(dx)$ [10]. Our next step is to generalize exactly solvable chaos (1) to capture the full domain of Lévy's stable laws. Now let us consider the mapping

$$X_{n+1}=\left|\tfrac{1}{2}(|X_n|^{\alpha}-1/|X_n|^{\alpha})\right|^{1/\alpha}\mathrm{sgn}[(X_n-1/X_n)]\equiv f_\alpha(X_n), \tag{4}$$

where $0<\alpha<2$, and $\mathrm{sgn}(x)=1$ for $x>0$ and $\mathrm{sgn}(x)=-1$ for $x<0$. We prove here that this chaotic dynamics (4) also has a mixing property similar to mapping (1), as well as an exact invariant density function given by

$$\rho_\alpha(x)=\frac{\alpha}{\pi}\frac{|x|^{\alpha-1}}{(1+|x|^{2\alpha})}\simeq\frac{\alpha}{\pi}|x|^{-(\alpha+1)}\quad\text{for}\quad |x|\to\infty. \tag{5}$$

Note that the chaotic system given by Eq. (1) is a special case of Eq. (4) with $\alpha=1$. We remark here that this system can also be seen as a doubling formula $s(2\theta)=f_\alpha[s(\theta)]$, where

$$s(\theta)=-\frac{\mathrm{sgn}\left[\tan\left(\frac{\pi}{2}\theta\right)\right]}{\left|\tan\left(\frac{\pi}{2}\theta\right)\right|^{1/\alpha}}.$$

Using the relations

$$s(2\theta)=f_\alpha[s(\theta)]\quad\text{for}\quad\theta\in[0,1),$$
$$s(2\theta-2)=f_\alpha[s(\theta)]\quad\text{for}\quad\theta\in[1,2] \tag{6}$$

and defining the diffeomorphism

$$x=\phi_\alpha^{-1}(\theta)=-\frac{\mathrm{sgn}\left[\tan\left(\frac{\pi}{2}\theta\right)\right]}{\left|\tan\left(\frac{\pi}{2}\theta\right)\right|^{1/\alpha}} \tag{7}$$

of $\theta\in[0,2]$ into $]-\infty,+\infty[$, we obtain the piecewise-linear map $g^{(2)}(\theta)=\phi_\alpha\cdot f_\alpha\cdot\phi_\alpha^{-1}(\theta)$ as

$$g^{(2)}(\theta)=\begin{cases}2\theta, & \theta\in[0,1)\\ 2\theta-2, & \theta\in[1,2]\end{cases} \tag{8}$$

with the invariant measure $\frac{1}{2}d\theta$ of $[0,2]$. Thus, the map of Eq. (4) preserves

$$\mu(dx)=\rho_\alpha(x)dx=\frac{1}{2}\frac{d\phi_\alpha}{dx}dx=\frac{\alpha}{\pi}\frac{|x|^{\alpha-1}}{(1+|x|^{2\alpha})}dx. \tag{9}$$

Therefore, the class of dynamical systems (4) with the parameter α also has a mixing property (thus is ergodic) with the Lyapunov exponent $\ln 2$.

More generally, from the family of Chebyshev maps $Y_{n+1}=f(Y_n)$ defined by the addition formulas of the form $\sin^2(p\theta)=f[\sin^2(\theta)]$, where $p=2,3,\ldots$, with the unique density $\sigma(y)=[1/\pi\sqrt{y(1-y)}]$ of the logistic map $Y_{n+1}=4Y_n(1-Y_n)$ (which corresponds to the case $p=2$) [11,12], we may construct infinitely many chaotic dynamical systems $X_{n+1}=f_\alpha(X_n)$ with the unique density function $\rho_\alpha(x)$ given by Eq. (5) [13]. For example, an explicit mapping with the Lyapunov exponent $\ln 3$ is given by

$$X_{n+1}=f_\alpha(X_n)=\left|\frac{|X_n|^\alpha(|X_n|^{2\alpha}-3)}{(3|X_n|^{2\alpha}-1)}\right|^{1/\alpha}\mathrm{sgn}\left[\frac{X_n(|X_n|^{2\alpha}-3)}{(3|X_n|^{2\alpha}-1)}\right], \tag{10}$$

which has the density (5) from the triplication formula of $s(\theta)$.

In this case, the topological conjugacy relation $g^{(3)}(\theta)=\phi_\alpha\circ f_\alpha\circ\phi_\alpha^{-1}(\theta)$ yields the piecewise-linear map

$$g^{(3)}(\theta)=\begin{cases}3\theta, & \theta\in\left[0,\frac{2}{3}\right)\\ 3\theta-2, & \theta\in\left[\frac{2}{3},\frac{4}{3}\right)\\ 3\theta-4, & \theta\in\left[\frac{4}{3},2\right].\end{cases} \tag{11}$$

In general, the same kind of topological conjugacy relation $g^{(p)}(\theta)=\phi_\alpha\circ f_\alpha\circ\phi_\alpha^{-1}(\theta)$ holds for a p-to-one piecewise-linear mapping $g^{(p)}(\theta)$. Let us consider slightly modified dynamical systems $X_{n+1}=f_{\alpha,\delta}(X_n)\equiv(1/\delta)f_\alpha(\delta X_n)$, with a change of variable $h(x)\equiv\delta x$ for a constant $\delta>0$. Thus this modified dynamics,

$$f_{\alpha,\delta}(X)=\left|\frac{1}{2}(X|^\alpha-1/|\delta^2X|^\alpha)\right|^{1/\alpha}\mathrm{sgn}\left[X-\frac{1}{\delta^2X}\right], \tag{12}$$

has an invariant measure $\rho_{\alpha,\delta}(x)dx=\delta\rho_\alpha(\delta x)dx=(\alpha\delta^\alpha|x|^{\alpha-1}dx)/\pi(1+\delta^{2\alpha}|x|^{2\alpha})$ with a slightly modified power-law tail given by

$$\rho_{\alpha,\delta}(x)\simeq\frac{\alpha}{\pi\delta^\alpha}\frac{1}{|x|^{\alpha+1}}\quad\text{for}\quad x\to\pm\infty. \tag{13}$$

We will show that this power-law tail of the density is sufficient for generating arbitrary symmetric stable laws. The canonical representation of stable laws obtained by Lévy and Khintchine [14,15] is

$$P(x;\alpha,\beta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(izx)\psi(z)dz, \tag{14}$$

where the characteristic function $\psi(z)$ is given by

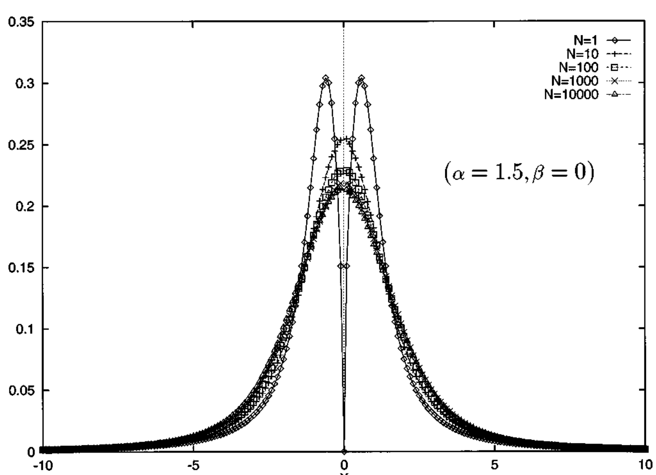

FIG. 1. Densities of the superposition $S_N=[\Sigma_{i=1}^N X(i)-A_N]/B_N$ of dynamical variables $X(i)$ generated by chaotic systems $X_{j+1}(i)=f_{\alpha=1.5}^{(2)}\times[X_j(i)]$, with N different initial conditions $X_0(i)|_{i=1,\cdots,N}$, are plotted for $N=1,10,100,1000$, and 10 000. In this case, the limit density converges to the symmetric Lévy's stable law with the indices $\alpha=1.5$ and $\beta=0$.

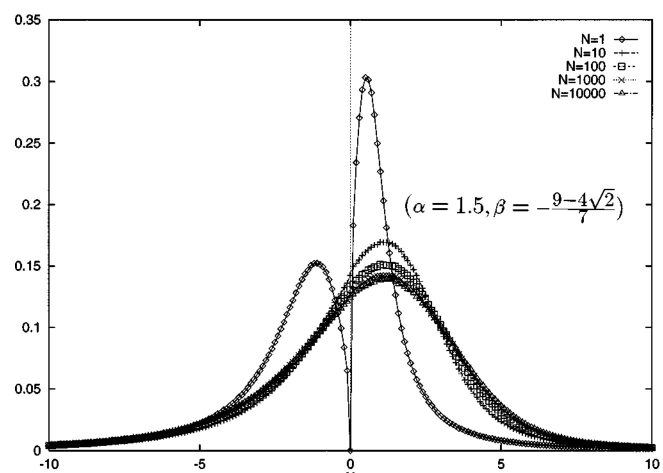

FIG. 2. Densities of the superposition $S_N=[\Sigma_{i=1}^N X(i)-A_N]/B_N$ of dynamical variables $X(i)$ generated by chaotic systems $X_{j+1}(i)=f_{\alpha=1.5,\delta_1=1,\delta_2=0.5}^{(2)}[X_j(i)]$, with N different initial conditions $X_0(i)|_{i=1,\ldots,N}$, are plotted for $N=1$, 10, 100, 1000, and 10 000. In this case, the limit density converges to the asymmetric Lévy's stable law with the indices $\alpha=1.5$ and $\beta=-[(9-4\sqrt{2})/7]$.

$$\psi(z)=\exp\{-i\gamma z-\eta|z|^{\alpha}[1+i\beta\ \mathrm{sgn}(z)\omega(z,\alpha)]\}, \tag{15}$$

α, β, γ, and η being real constants satisfying $0<\alpha\leqslant 2, -1\leqslant\beta\leqslant 1, \gamma\geqslant 0$, and

$$\omega(z,\alpha)=\tan(\pi\alpha/2)\quad \text{for}\quad \alpha\neq 1,$$

$$\omega(z,\alpha)=(2/\pi)\ln|z|\quad \text{for}\quad \alpha=1. \tag{16}$$

Note that explicit forms of Lévy's distributions (14) are not known for general parameters α and β except for a few cases such as the Cauchy distribution ($\alpha=1,\beta=0$). According to the generalized central limit theorem (GCLT), it is known [16] that if the density function of a stochastic process has a long tail,

$$\rho(x)\simeq c_-|x|^{-(1+\alpha)}\quad \text{for}\quad x\to-\infty,$$

$$\rho(x)\simeq c_+|x|^{-(1+\alpha)}\quad \text{for}\quad x\to+\infty, \tag{17}$$

then the superposition $S_N=[\Sigma_{i=1}^{N}X(i)-A_N]/B_N$ of independent, identically distributed random variables $X(1),\ldots,X(N)$ with the density $\rho(x)$ converges in density to a Lévy's stable law $P(x;\alpha,\beta)$, with

$$\beta=(c_+-c_-)/(c_++c_-),$$

$$A_N=0,\quad B_N=N^{1/\alpha},$$

$$\eta=\frac{\pi(c_++c_-)}{2\alpha\sin(\pi\alpha/2)\Gamma(\alpha)}\quad \text{for}\quad 0<\alpha<1, \tag{18}$$

$$A_N=N\langle x\rangle,\quad B_N=N^{1/\alpha},$$

$$\eta=\frac{\pi(c_++c_-)}{2\alpha^2\sin(\pi\alpha/2)\Gamma(\alpha-1)}\quad \text{for}\quad 1<\alpha<2.$$

In the case of the symmetric long tail of Eq. (5), $c_+=c_-=(\alpha/\pi\delta^{\alpha})$, $\beta=0$, and η is determined by

$$\eta=\frac{1}{\delta^{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha)}\quad \text{for}\quad 0<\alpha<1,$$

$$\eta=\frac{1}{\alpha\delta^{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha-1)}\quad \text{for}\quad 1<\alpha<2. \tag{19}$$

Thus, according to the GCLT, the superposition of statistically independent, identically distributed random variables generated by N chaotic systems of Eq. (4) is guaranteed to converge in distribution to an arbitrary symmetric Lévy's stable law $P(x;\alpha,\beta=0)$. Figure 1 shows that the convergence in distribution to a Lévy's stable distribution with parameters $\alpha=1.5$ and $\beta=0$, as predicted by the GCLT, is clearly seen for $N=10\,000$. Similar results hold for more general stable distributions, including asymmetric stable laws. In particular, let us consider a family of dynamical systems $X_{n+1}=f_{\alpha,\delta_1,\delta_2}(X_n)$, where

$$f_{\alpha,\delta_1,\delta_2}(X)=\begin{cases}\frac{1}{\delta_1^2|X|}\left(\frac{|\delta_1X|^{2\alpha}-1}{2}\right)^{1/\alpha} & \text{for}\quad X>\frac{1}{\delta_1}\\ -\frac{1}{\delta_1\delta_2|X|}\left(\frac{1-|\delta_1X|^{2\alpha}}{2}\right)^{1/\alpha} & \text{for}\quad 0<X<\frac{1}{\delta_1}\\ \frac{1}{\delta_1\delta_2|X|}\left(\frac{1-|\delta_2X|^{2\alpha}}{2}\right)^{1/\alpha} & \text{for}\quad -\frac{1}{\delta_2}<X<0\\ -\frac{1}{\delta_2^2|X|}\left(\frac{|\delta_2X|^{2\alpha}-1}{2}\right)^{1/\alpha} & \text{for}\quad X<-\frac{1}{\delta_2}.\end{cases} \tag{20}$$

We can show that this dynamical system $X_{n+1}=f_{\alpha,\delta_1,\delta_2}(X_n)$ has the asymmetric invariant measure $\mu(dx)=\rho(x;\alpha,\delta_1,\delta_2)dx$, where $\rho_{\alpha,\delta_1,\delta_2}(x)$ is given by

$$\rho_{\alpha,\delta_1,\delta_2}(x)=\frac{\alpha\delta_1^{\alpha}x^{\alpha-1}}{\pi(1+\delta_1^{2\alpha}x^{2\alpha})}\quad \text{for}\quad x>0,$$

$$\rho_{\alpha,\delta_1,\delta_2}(x)=\frac{\alpha\delta_2^{\alpha}|x|^{\alpha-1}}{\pi(1+\delta_2^{2\alpha}|x|^{2\alpha})}\quad \text{for}\quad x<0. \tag{21}$$

Because the power-law tail is asymmetric as

$$\rho_{\alpha,\delta_1,\delta_2}(x)\simeq\frac{\alpha}{\pi\delta_1^{\alpha}x^{\alpha+1}},\quad x\to+\infty,$$

$$\rho_{\alpha,\delta_1,\delta_2}(x)\simeq\frac{\alpha}{\pi\delta_2^{\alpha}|x|^{\alpha+1}},\quad x\to-\infty \tag{22}$$

for $\delta_1\neq\delta_2$, the GCLT guarantees that the limiting distribution would be a Lévy's canonical form $P(x;\alpha,\beta)$ with the skewness parameter $\beta=[(\delta_2^\alpha-\delta_1^\alpha)/(\delta_1^\alpha+\delta_2^\alpha)]\neq 0$. Thus, one can generate an arbitrary Lévy's stable laws $P(x;\alpha,\beta)$ for $0<\alpha<2$ and $-1\leqslant\beta\leqslant 1$ [17] using the chaotic mappings $f_{\alpha,\delta_1,\delta_2}(X)$ with proper parameters α, δ_1, and δ_2. Figure 2 illustrates convergence to an asymmetric Lévy's stable distribution with indices $\alpha=1.5$ and $\beta=-[(9-4\sqrt{2})/7]$ as clearly seen for $N=10\,000$, as predicted by the GCLT. To show the exactness of the asymmetric density (21), we must check that the invariant measure $\rho_{\alpha,\delta_1,\delta_2}(x)dx$ satisfies the probability preservation relation (Perron-Frobenius equation) [18]

$$\rho_{\alpha,\delta_1,\delta_2}(y)=\sum_{x=f^{-1}_{\alpha,\delta_1,\delta_2}(y)}\rho_{\alpha,\delta_1,\delta_2}(x)\left|\frac{dx}{dy}\right|. \tag{23}$$

We note that $f_{\alpha,\delta,\delta}(x)=f_{\alpha,\delta}(x)$ and $\rho_{\alpha,\delta_1,\delta_2}(x)=\rho_{\alpha,\delta_1}(x)$ for $x>0$ and $\rho_{\alpha,\delta_1,\delta_2}(x)=\rho_{\alpha,\delta_2}(x)$ for $x<0$, which also have the Perron-Frobenius equations

$$\rho_{\alpha,\delta_i}(y)=\sum_{x=f^{-1}_{\alpha,\delta_i}(y)}\rho_{\alpha,\delta_i}(x)\left|\frac{dx}{dy}\right| \quad \text{for} \quad i=1,2. \tag{24}$$

Here, we define two preimages $x_a=f^{-1}_{\alpha,\delta_1,\delta_2}(y)<0$ and $x_b=f^{-1}_{\alpha,\delta_1,\delta_2}(y)>0$ for $y>0$. In the case $f_{\alpha,\delta_1}(x)=y>0$, we also define two preimages $x'_a=f^{-1}_{\alpha,\delta_1}(y)<0$ and $x'_b=f^{-1}_{\alpha,\delta_1}(y)(=x_b)>0$ for $y=f_{\alpha,\delta_1}(x)>0$. It is easy to check that $\delta_2 x_a=\delta_1 x'_a$. From the Perron-Frobenius equations (23) and (24), we have the relation

$$\rho_{\alpha,\delta_2}(x_a)\frac{1}{\left|\frac{df_{\alpha,\delta_1,\delta_2}(x)}{dx}\right|_{x=x_a}}=\rho_{\alpha,\delta_1}(x'_a)\frac{1}{\left|\frac{df_{\alpha,\delta_1}(x)}{dx}\right|_{x=x'_a}}. \tag{25}$$

We note that the validity of Eq. (25) can be checked under the condition $\delta_2 x_a=\delta_1 x'_a$. In the same manner, we can also show that $\rho_{\alpha,\delta_1,\delta_2}(y)$ satisfies the Perron-Frobenius equation (23) for $y<0$.

There is also an interesting dualistic structure in these types of chaotic dynamical systems. Let us consider dynamical systems $X^*_{n+1}=f^*_{\alpha,\delta_1,\delta_2}(X^*_n)$ defined as

$$\frac{1}{f_{\alpha,1/\delta_1,1/\delta_2}(1/X^*_n)}.$$

Because the normalized and symmetric case, $f^*_\alpha(X^*)\equiv f^*_{\alpha,\delta_1=1,\delta_2=1}(X^*)$, can also be seen as the doubling formula of $-\mathrm{sgn}[\tan((\pi/2)\theta)]|\tan((\pi/2)\theta)|^{1/\alpha}$, and the piecewise-linear map $g^{(2)}(\theta)=\phi^*_\alpha\circ f^*_\alpha\circ\phi^{*-1}_\alpha(\theta)$ [Eq. (2)] can be derived by the diffeomorphism $x^*=\phi^{*-1}_\alpha(\theta)=-\mathrm{sgn}[\tan((\pi/2)\theta)]|\tan((\pi/2)\theta)|^{1/\alpha}$, map $f^*_\alpha(X^*)$ also has the invariant measure $\rho_\alpha(x)$ of Eq. (5). Thus in the same way as that used in obtaining $\rho_{\alpha,\delta_1,\delta_2}(x)$ for $f_{\alpha,\delta_1,\delta_2}(X)$, we can show that this dual map has the invariant measure $\rho_{\alpha,1/\delta_1,1/\delta_2}(x)dx$. This dualistic structure of the dynamical systems $f_{\alpha,\delta_1,\delta_2}(X)$ and $f^*_{\alpha,1/\delta_1,1/\delta_2}(X^*)$ originates from the relation $\phi^{-1}_\alpha(\theta)\cdot\phi^{*-1}_\alpha(\theta)=1$.

In summary, we have found that Lévy's stable laws can be directly generated by the superposition of many independent, identically distributed dynamical variables obeying certain chaotic processes in a unified manner. Owing to the ubiquitous and generally implicit character of Lévy's stable laws, our explicit implementations using ergodic transformations with long-tail densities have a potentially broad range of applications to many different physical problems.

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[1] J. Holtzmark, Ann. Phys. (Leipzig) **58**, 577 (1919); S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[2] B. B. Mandelbrot, J. Business **36**, 394 (1963).
[3] R. N. Mantegna and H. E. Stanley, Nature (London) **376**, 46 (1995).
[4] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[5] P. Manneville, J. Phys. (France) **41**, 1235 (1980).
[6] T. Geisel and S. Thomae, Phys. Rev. Lett. **52**, 1936 (1984).
[7] B. D. Hughes, M. F. Shlesinger, and E. W. Montroll, Proc. Natl. Acad. Sci. USA **78**, 3287 (1981).
[8] L. Devroye, *Non-Uniform Random Variate Generation* (Springer-Verlag, New York, 1986).
[9] R. N. Mantegna, Phys. Rev. E **49**, 4677 (1994).
[10] J. P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).
[11] S. M. Ulam and J. von Neumann, Bull. Am. Math. Soc. **53**, 1120 (1947).
[12] R. L. Adler and T. J. Rivlin, Proc. Am. Math. Soc. **15**, 794 (1964).
[13] K. Umeno, Phys. Rev. E **55**, 5280 (1997).
[14] A. Ya. Khintchine and P. Lévy, C. R. Acad. Sci. Paris **202**, 374 (1936).
[15] P. Lévy, *Théorie de l'Addition des Variables Aléatories* (Gauthier-Villars, Paris, 1937).
[16] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Reading, MA, 1954).
[17] The cases for $\beta=\pm 1$ correspond to the purely asymmetric stable distributions over the semi-infinite support $I_+=(0,\infty)$ for $\beta=1$, and $I_-=(-\infty,0)$ for $\beta=-1$. These completely asymmetric stable distributions can be generated by the chaotic systems $X_{n+1}=|f_{\alpha,\delta}(X_n)|$ with the exact densities $\rho^+_\alpha(x)=(2\alpha/\pi)[x^{\alpha-1}/(1+x^{2\alpha})]$ on I_+ for $\beta=1$ and $X_{n+1}=-|f_{\alpha,\delta}(X_n)|$ with the density $\rho^-_\alpha(x)=(2\alpha/\pi)[|x|^{\alpha-1}/(1+|x|^{2\alpha})]$ on I_- for $\beta=-1$.
[18] A. Lasota and M. C. Mackey, *Chaos, Fractals and Noise*, 2nd ed. (Springer-Verlag, Berlin, 1994); A. Boyarsky and P. Góra, *Laws of Chaos* (Birkhäuser, Boston, 1997).